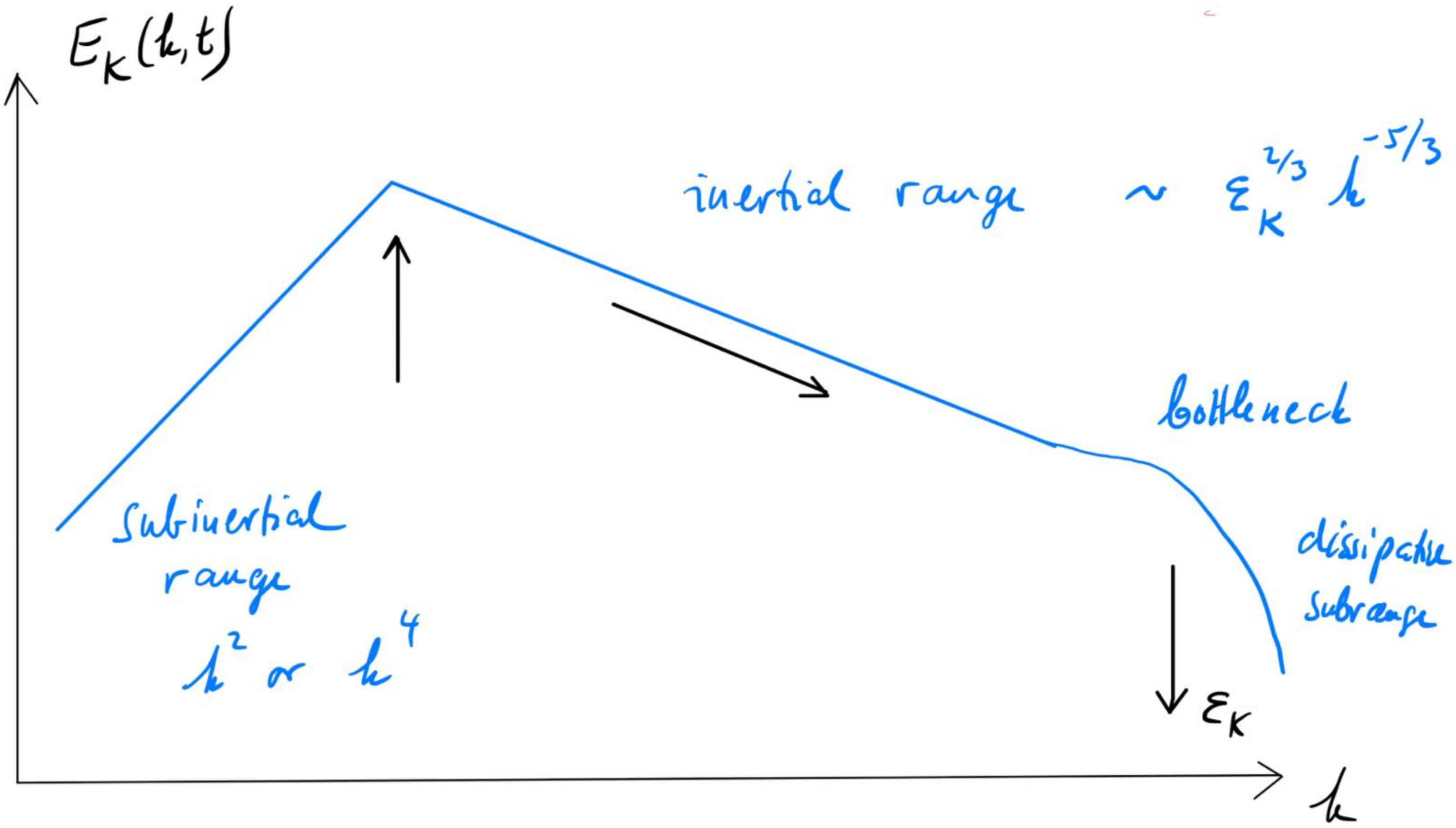
Turbulent magnetic fields: cascades & dissipation

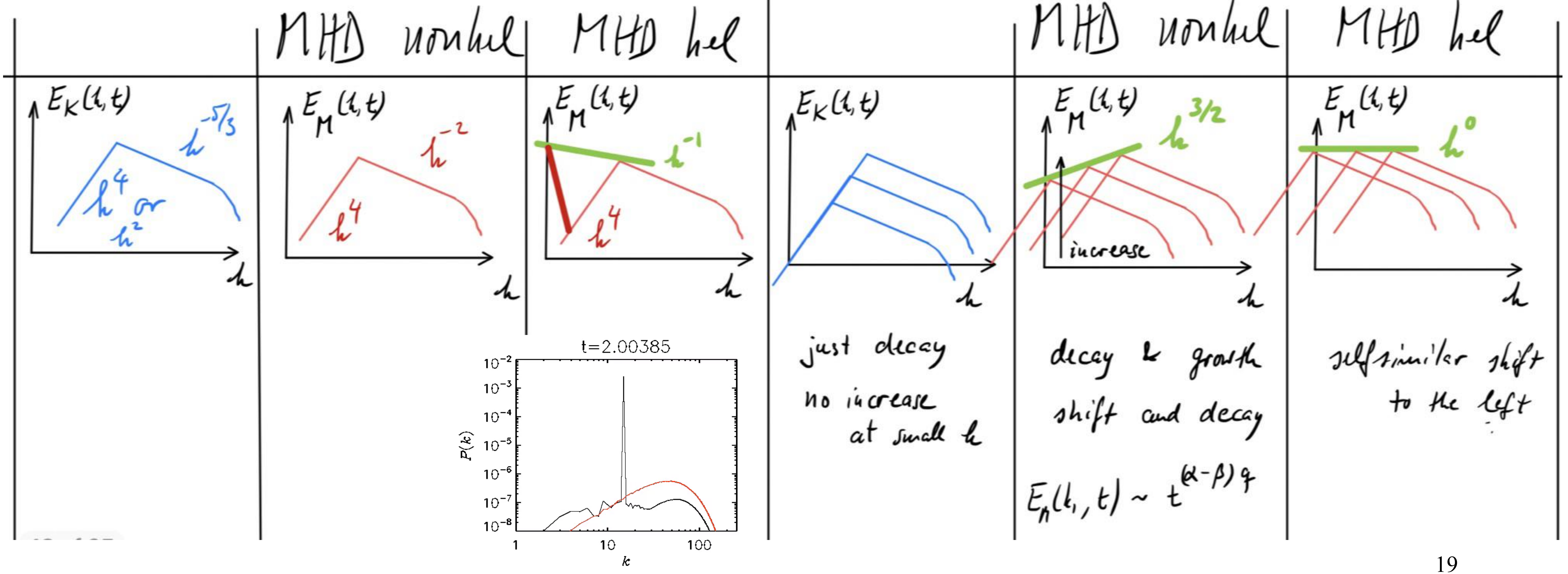
- Turbulence spectrum important diagnostics
  - Spectral slope (inertial range)
  - Dimensional arguments
  - Subinertial range
  - Random (δ-correlated): k^2, because integrated over shells in k-space
- Length of inertial range
  - → Reynolds number
  - Very large in astrophysics
  - Not in simulations
  - Some effects sensitive to this
- Bottleneck effect
  - Is a real effect
  - Less pronounced in 1-D spectra
  - Important for some small-scale dynamos



Cascades (periodic box)

forced turbulence

decaying turbulence



The conclusion from the above expressions is thus that *the MHD equations in an expanding universe with zero curvature are the same as the relativistic MHD equations in a nonexpanding universe, provided the dynamical quantities are replaced by the scaled "tilde" variables, and provided conformal time $\tilde{t}$ is used.* The effect of this is, as usual, that

3-D decay simulations

Brandenburg, Enqvist, Olesen
Phys Rev D 54, 1291 (1996)

Initial slope

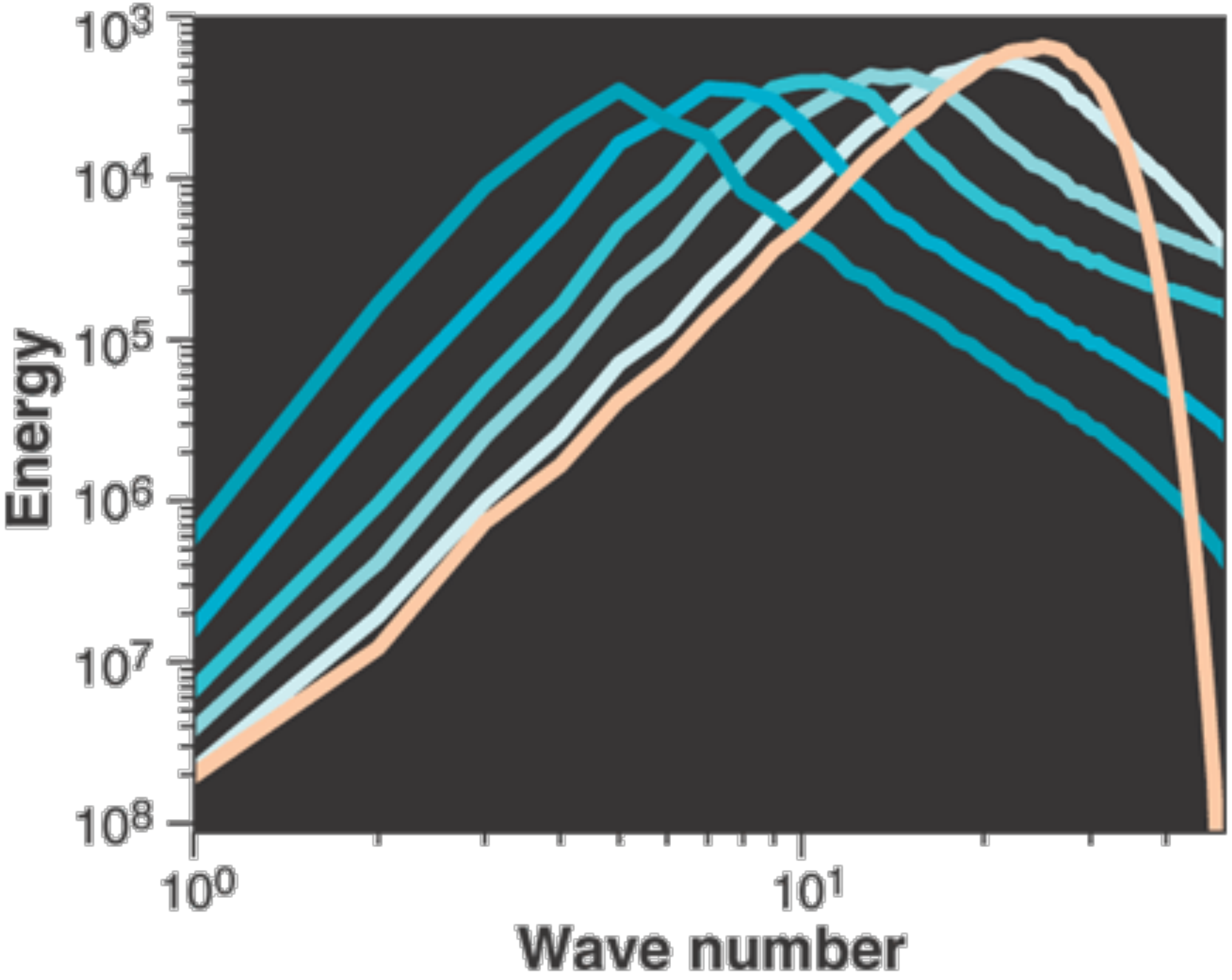
$E{\sim}k^4$

Causality (Durrer & Caprini 2003)
shell-integrated spectra
δ-correlated vector potential

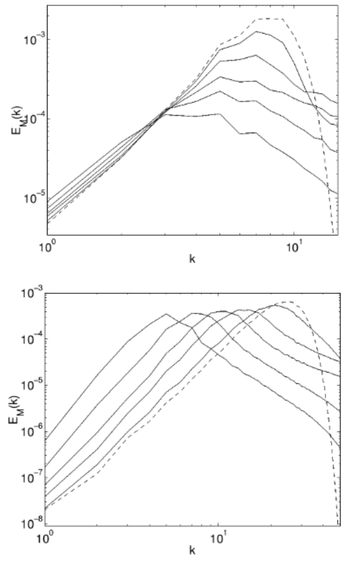
$$E_{\rm M}(k) = \frac{1}{2} \sum_{k_- < |\boldsymbol{k}| \leqslant k_+} |\tilde{\boldsymbol{B}}(\boldsymbol{k})|^2,$$

$$H_{\rm M}(k) = \frac{1}{2} \sum_{k_- < |\boldsymbol{k}| \leqslant k_+} (\tilde{\boldsymbol{A}} \cdot \tilde{\boldsymbol{B}}^* + \tilde{\boldsymbol{A}}^* \cdot \tilde{\boldsymbol{B}}),$$

$k_\pm = k \pm \delta k/2$ and $\delta k = 2\pi/L$ is the



helical vs
nonhelical



Christensson et al.
(2001, PRE 64, 056405)

Self-similar turbulent decay

$$E_{\rm M}(k\xi_{\rm M}(t),t) \approx \xi_{\rm M}^{-\beta_{\rm M}} \phi(k\xi_{\rm M}).$$

$$\int E_i(k,t)\,{\rm d}k = \mathcal{E}_i \qquad \xi_i(t) = \int_0^\infty k^{-1} E_i(k,t)\,{\rm d}k/\mathcal{E}_i(t)$$

instantaneous scaling exponents

$$p_i(t) = d\ln\mathcal{E}_i/d\ln t, \quad q_i(t) = d\ln\xi_i/d\ln t,$$

β	p	q	inv.	dim.
4	$10/7 \approx 1.43$	$2/7 \approx 0.286$	$\mathcal{L}$	$[x]^7[t]^{-2}$
3	$8/6 \approx 1.33$	$2/6 \approx 0.333$		
2	$6/5 = 1.20$	$2/5 = 0.400$		
1	$4/4 = 1.00$	$2/4 = 0.500$	$\langle \boldsymbol{A}_{\rm 2D}^2 \rangle$	$[x]^4[t]^{-2}$
0	$2/3 \approx 0.67$	$2/3 \approx 0.667$	$\langle \boldsymbol{A}\cdot\boldsymbol{B} \rangle$	$[x]^3[t]^{-2}$
-1	$0/2 = 0.00$	$2/1 = 1.000$		

→ 3/2

$$1 + \beta_{\rm M} = p_{\rm M}/q_{\rm M}$$

growth at small k

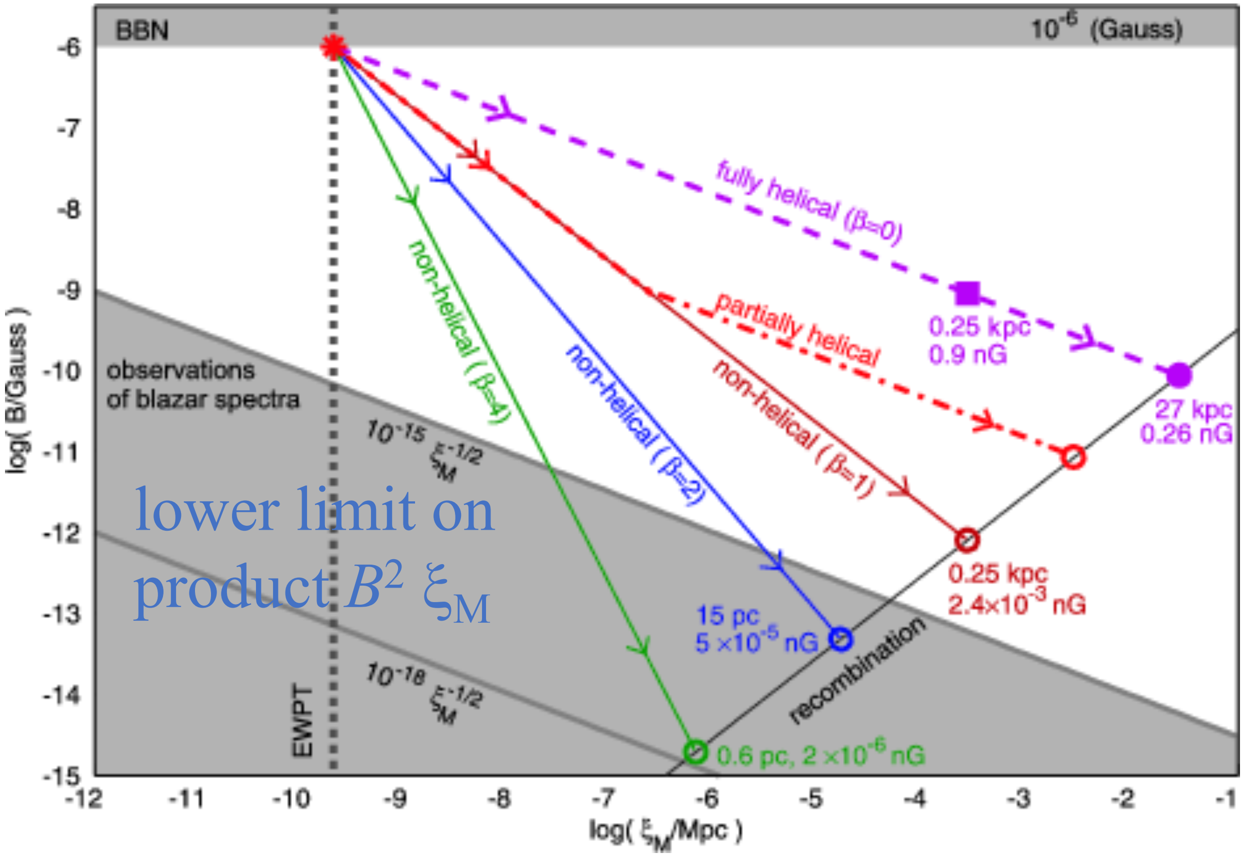
$$E_{\rm M}(k,t) = \xi_{\rm M}^{\alpha-\beta}\, k^\alpha \quad (k\xi_{\rm M} \ll 1)$$

α	β	q	$(\alpha-\beta)\,q$	comment, property
1.7	3/2	4/9	4/45	possible scaling, assuming Hosking integral conserved
1.7	2	2/5	$-3/25$ or 0	alternative scaling if Saffman integral conserved
2	2	2/5	0	Saffman scaling, assuming Saffman integral conserved
2	3/2	4/9	$2/9 \approx 0.22$	Saffman scaling, assuming Hosking integral conserved
3	2	2/5	$2/5 = 0.4$	cubic scaling, assuming Saffman integral conserved
3	3/2	4/9	$2/3 \approx 0.67$	cubic scaling, assuming Hosking integral conserved
4	3/2	4/9	$10/9 \approx 1.11$	Batchelor scaling, assuming Hosking integral conserved
4	0	2/3	$8/3 \simeq 2.7$	fully helical

Evolutionary diagram

Magnetic helicity	$\langle \boldsymbol{A}\cdot\boldsymbol{B}\rangle$	$\mathrm{cm}^3\,\mathrm{s}^{-2}$	$\xi_\mathrm{M}(t) \propto \langle \boldsymbol{A}\cdot\boldsymbol{B}\rangle^{1/3} t^{2/3}$
Anastrophy (2-D)	$\langle A_z^2\rangle$	$\mathrm{cm}^4\,\mathrm{s}^{-2}$	$\xi_\mathrm{M}(t) \propto \langle A_z^2\rangle^{1/4} t^{1/2}$
Hosking integral	I_H	$\mathrm{cm}^9\,\mathrm{s}^{-4}$	$\xi_\mathrm{M}(t) \propto I_\mathrm{H}^{1/9} t^{4/9}$
Saffman integral	I_S	$\mathrm{cm}^5\,\mathrm{s}^{-2}$	$\xi_\mathrm{M}(t) \propto I_\mathrm{S}^{1/5} t^{2/5}$
Loitsyansky integral	I_L	$\mathrm{cm}^7\,\mathrm{s}^{-2}$	$\xi_\mathrm{M}(t) \propto I_\mathrm{S}^{1/7} t^{2/7}$

AB, Kahniashvili, ..., Vachaspati (2017)

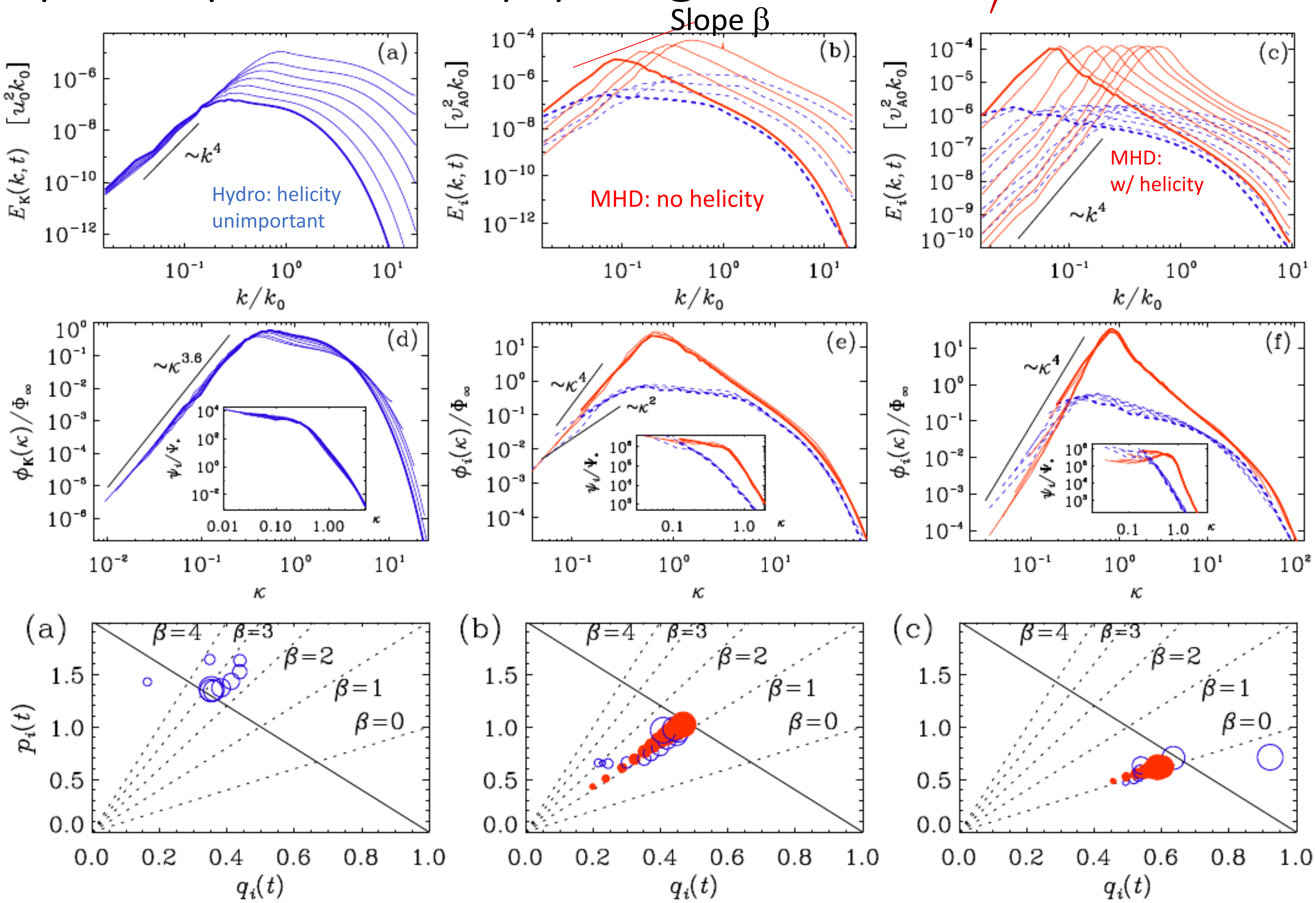


Magnetic energy dependence
Parametric representation

magnetic energy	$\kappa = p/2q$
$\mathcal{E}_\mathrm{M}(t) \propto \langle \boldsymbol{A}\cdot\boldsymbol{B}\rangle^{2/3} t^{-2/3}$	$\propto \xi_\mathrm{M}^{-1/2}$
$\mathcal{E}_\mathrm{M}(t) \propto \langle A_z^2\rangle^{1/2} t^{-1}$	$\propto \xi_\mathrm{M}^{-1}$
$\mathcal{E}_\mathrm{M}(t) \propto I_\mathrm{H}^{2/9} t^{-10/9}$	$\propto \xi_\mathrm{M}^{-5/4}$
$\mathcal{E}_\mathrm{M}(t) \propto I_\mathrm{S}^{2/5} t^{-6/5}$	$\propto \xi_\mathrm{M}^{-3/2}$
$\mathcal{E}_\mathrm{M}(t) \propto I_\mathrm{S}^{2/7} t^{-10/7}$	$\propto \xi_\mathrm{M}^{-5/2}$

Collapsed spectra and *pq* diagrams

$p_i(t) = \cancel{-}\,d\ln\mathcal{E}_i/d\ln t, \quad q_i(t) = d\ln\xi_i/d\ln t,$



Explanations for slope β

Exponents *p*,*q*

(Hosking & Schekochihin 2021+2023)

Brandenburg & Kahniashvili (2017)

Anastrophy in 2-D

$$\text{conservation of anastrophy, } \langle A_z^2 \rangle = \text{const}$$

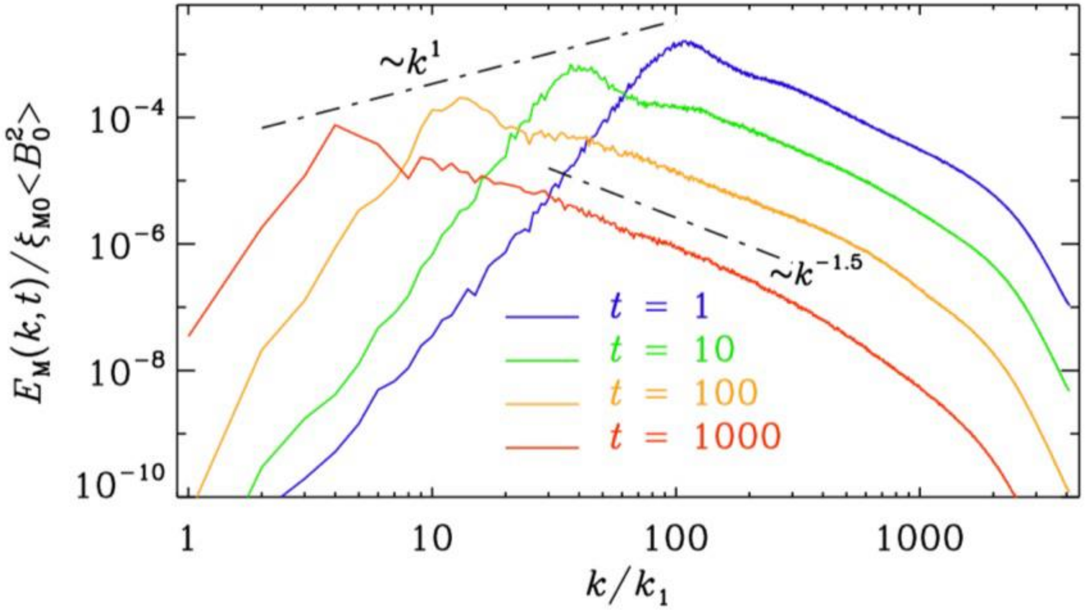
Vector potential (0, 0, A_z) obeys $\frac{\mathrm{D}A_z}{\mathrm{D}t} = \eta \nabla^2 A_z$

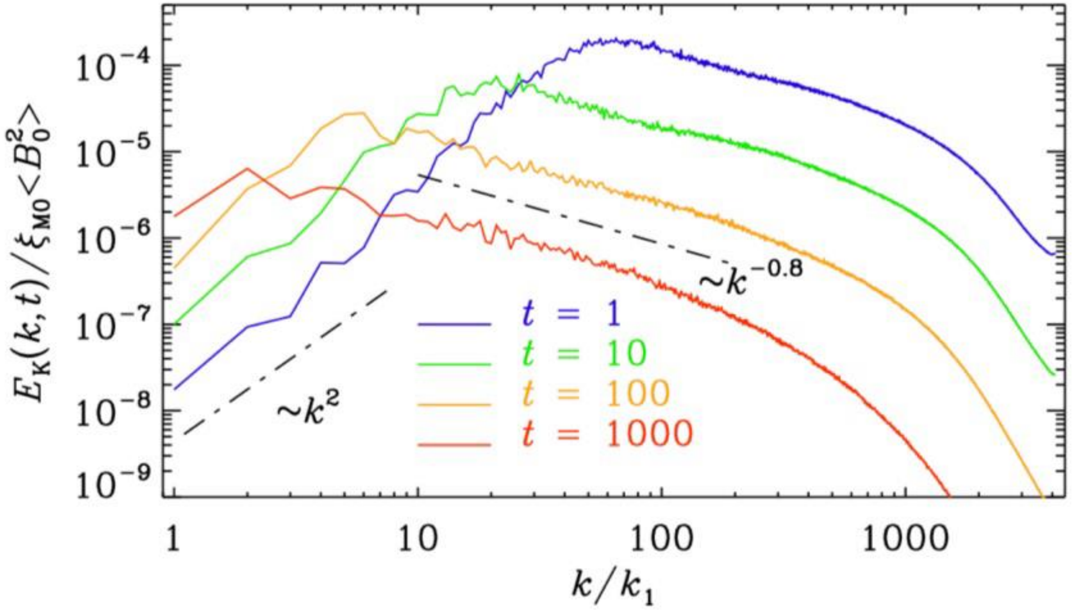
Slightly different decay laws

$$\xi_\mathrm{M}(t) \approx 0.13 \langle A_z^2 \rangle^{1/4} t^{1/2}, \quad \mathcal{E}_\mathrm{M}(t) \approx 15 \langle A_z^2 \rangle^{1/2} t^{-1}$$

Envelope: linear increase

$$E_\mathrm{M}(k,t) \le 60 \langle A_z^2 \rangle k$$





Appendix A: Historical note on anastrophy

In recent years, the term anastrophy for the mean squared magnetic vector potential $\langle A_z^2 \rangle$ = const has become increasingly popular (Tronko et al. 2013; Galtier & Meyrand 2015; Zhou et al. 2021; Hosking & Schekochihin 2021; Schekochihin 2022). In the 1970s, it was referred to as mean square vector potential (Fyfe & Montgomery 1976) or as the variance of the magnetic potential (Pouquet 1978). The term anastrophy was first used in the 1987 Les Houches lecture notes by Pouquet (1993), and it was also used by Vakoulenko (1993), but without explanation of its origin.

Annick Pouquet (private communication) informed us now that the word was invented by Uriel Frisch and Nicolas Papanicolaou during a meeting on a Winter Sunday in the 1970s at Saint-Jean-Cap-Ferrat, while she and Jacques Léorat were also present.

The word has Greek roots and refers to the rate or speed of something, but is not to be confused with the enstrophy, i.e., the mean squared vorticity.

The new kid in town...

Reconnection-Controlled Decay of Magnetohydrodynamic Turbulence and the Role of Invariants

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Scaling of the Hosking integral in decaying magnetically dominated turbulence

Hongzhe Zhou [1,2,†], Ramkishor Sharma[1,3] and Axel Brandenburg [1,3,4,5]

[1]Nordita, KTH Royal Institute of Technology and Stockholm University, Hannes Alfvéns väg 12, SE-10691 Stockholm, Sweden

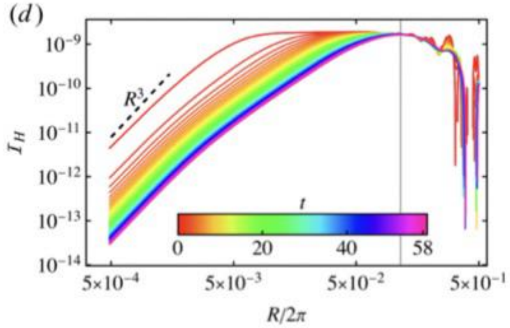
$$h(\boldsymbol{x}, t) = \boldsymbol{A} \cdot \boldsymbol{B}$$

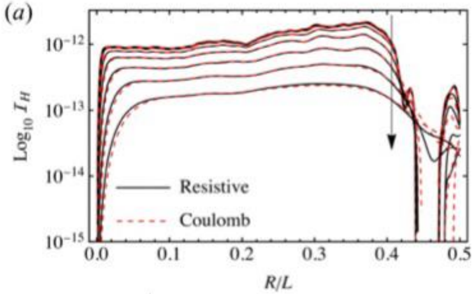
$\vec{B} = \nabla \times \vec{A}$
gauge dependent

$$\mathcal{I}_H(R) = \int_{V_R} \mathrm{d}^3 r \, \langle h(\boldsymbol{x}) h(\boldsymbol{x} + \boldsymbol{r}) \rangle$$

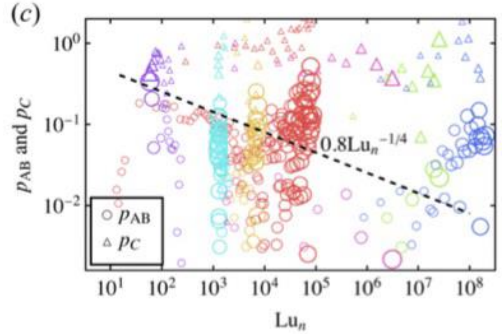
$$I_H = \mathcal{I}_H(\xi_M \ll R \ll L) = \frac{1}{V_R} \langle H_{V_R}^2 \rangle$$

$$H_{V_R} = \int_{V_R} \mathrm{d}^3 r \, h(\boldsymbol{r}).$$





↳ gauge invariant



Lundquist	rate
10^3	0.1
10^7 (hyper)	0.01

⇒ conserved

† We are grateful to Keith Moffatt for alerting us to the fact that the formerly used term "Saffman helicity invariant" may be misleading, because the term helicity invariant is reserved for integrals which are chiral in character. Moreover, Saffman never considered helicity in his papers. The term "magnetic helicity density correlation integral" may be more appropriate but rather clumsy. Following Schekochihin (2020), we now refer to it as the Hosking integral. We use the term integral instead of invariant as long as we are not in the ideal limit.

Hosking integral

$$\mathcal{I}_H(R \ll \xi_M) \simeq \int_{V_R} \mathrm{d}^3 r \, \langle h(\boldsymbol{x}) h(\boldsymbol{x}) \rangle \propto R^3$$

$$\mathcal{I}_H(R) = \int_0^\infty \mathrm{d}k \, w_{\mathrm{sph}}^{\mathrm{BC}}(k) \, \mathrm{Sp}\,(h)$$

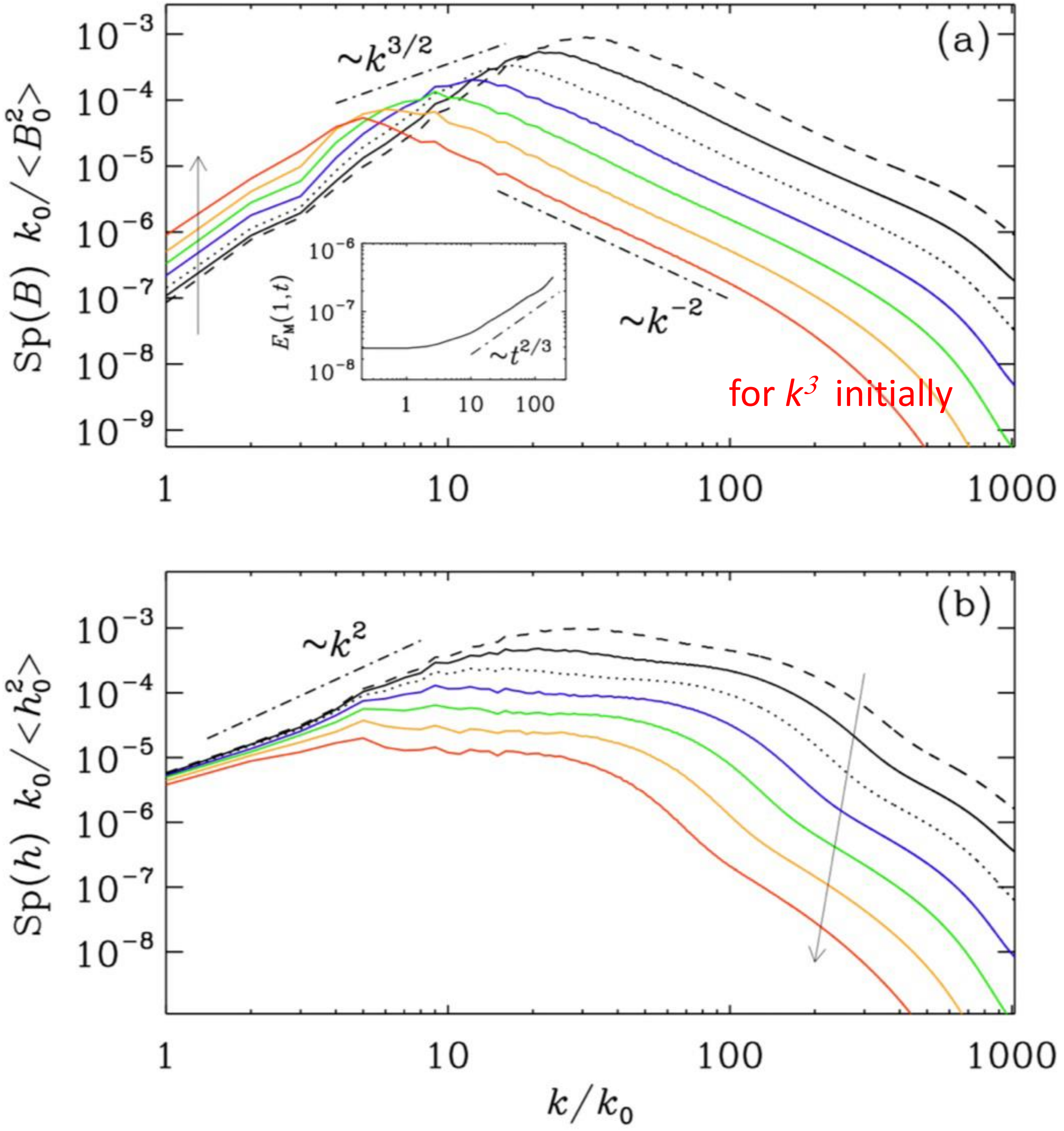
$$\mathrm{Sp}\,(h) = \frac{1}{V} \frac{k^2}{(2\pi)^3} \int_{|\boldsymbol{k}|=k} \mathrm{d}\Omega_k \, \tilde{h}^*(\boldsymbol{k}) \tilde{h}(\boldsymbol{k})$$

$$\mathrm{Sp}\,(h) = \frac{I_H}{2\pi^2} k^2 + \mathcal{O}(k^4)$$

$[I_\mathrm{H}] = \mathrm{cm}^9 \ \mathrm{s}^{-4}$

$\xi_M = I_\mathrm{H}^{\,a} \ t^{-b}$

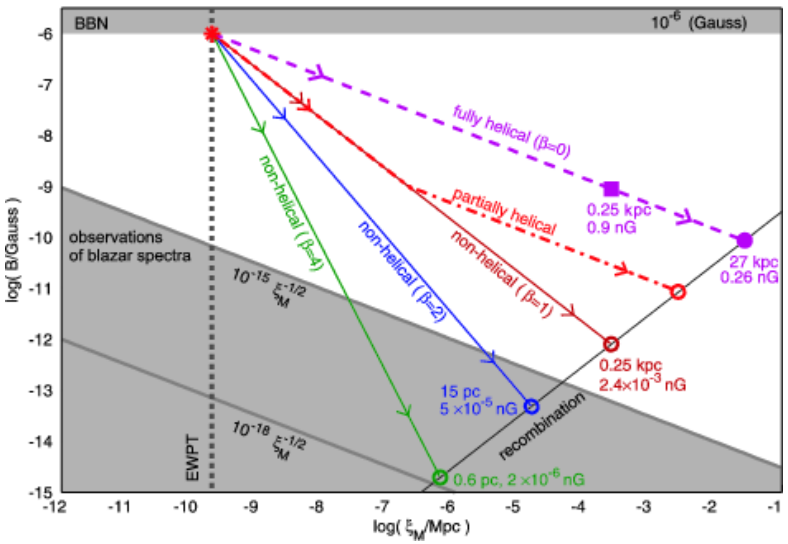
a=1/9, b=4/9



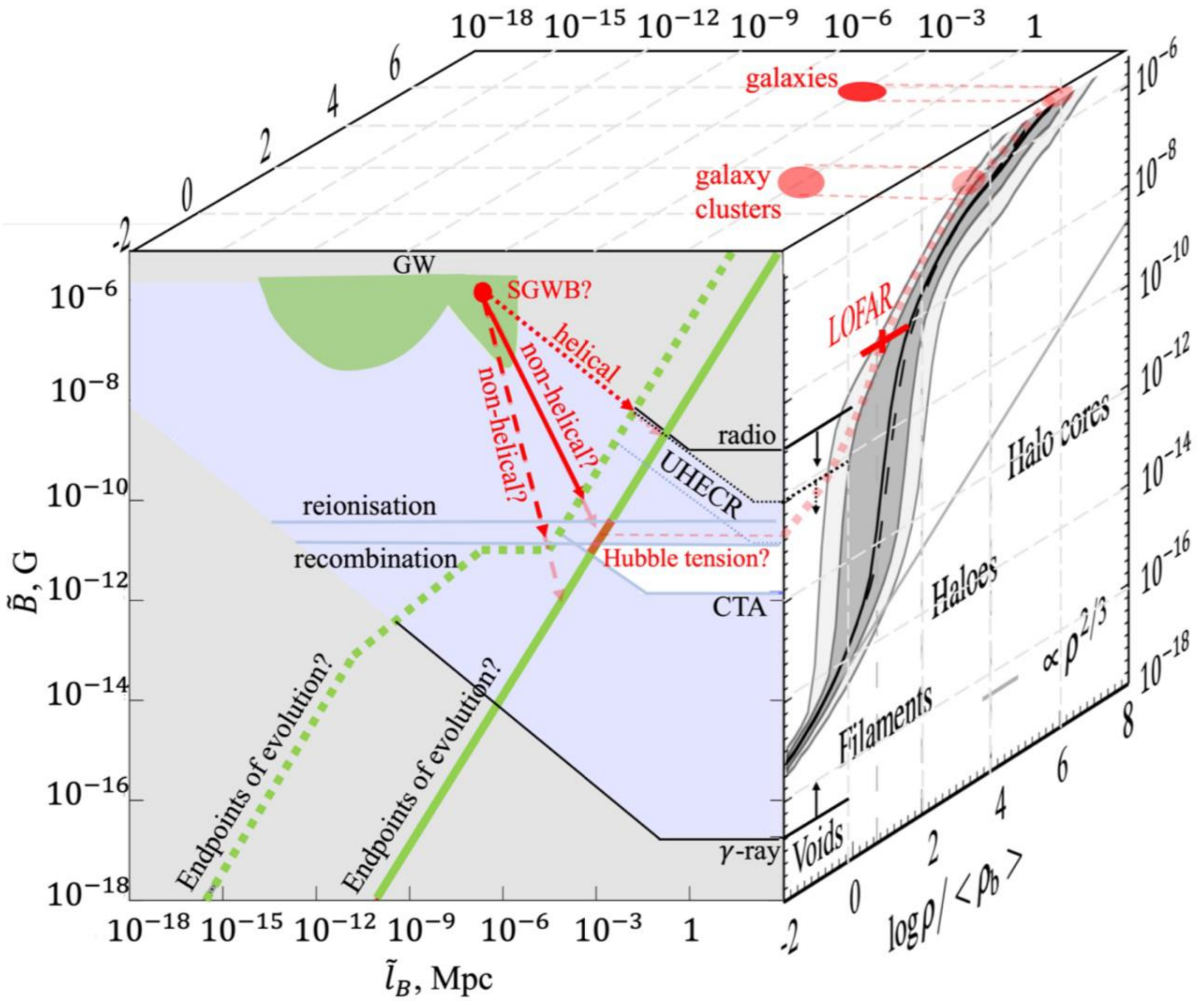
AB, Sharma, Vachaspati (2023)

$$\xi_\mathrm{M}(t) \approx 0.12 \, I_\mathrm{H}^{1/9} t^{4/9}, \quad \mathcal{E}_\mathrm{M}(t) \approx 3.7 \, I_\mathrm{H}^{2/9} t^{-10/9}, \quad E_\mathrm{M}(k,t) \lesssim 0.025 \, I_\mathrm{H}^{1/2} (k/k_0)^{3/2}$$

Resistively prolonged decay during radiative era



- Endpoints under assumption that decay time = Alfven time
- Use: decay time = recombination time
- Possibility: decay time >> Alfven time
- → Premature endpoint of evolution



Resistively controlled primordial magnetic turbulence decay

A. Brandenburg[1,2,3,4,5], A. Neronov[6,7], and F. Vazza[8,9,10]

Relation between decay time

$$\tau^{-1} = -\mathrm{d}\ln\mathcal{E}_{\mathrm{M}}/\mathrm{d}t$$

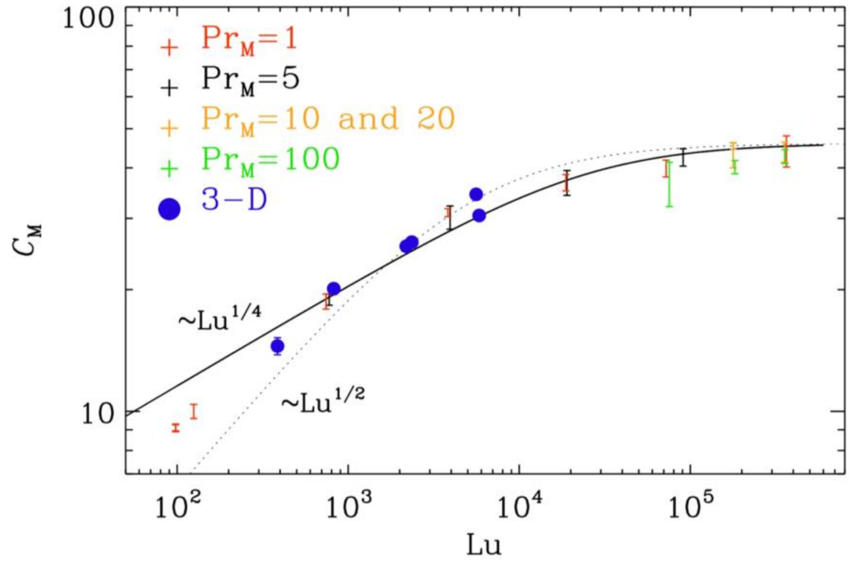
and Alfven time

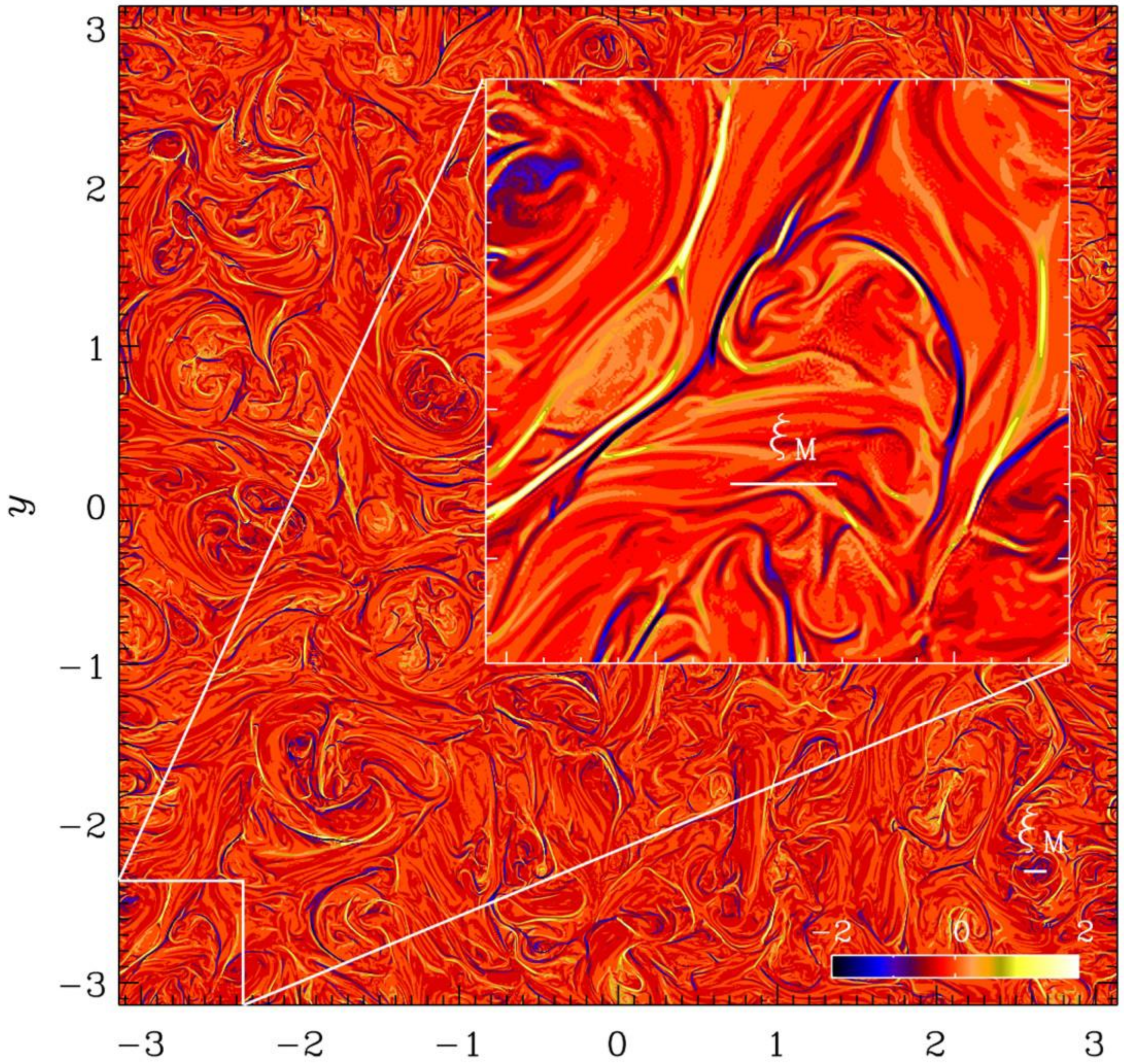
$$\tau_{\mathrm{A}} = \xi_{\mathrm{M}}/v_{\mathrm{A}}$$

$$\mathcal{E}_{\mathrm{M}} = B_{\mathrm{rms}}^2/2\mu_0 = \rho v_{\mathrm{A}}^2/2$$

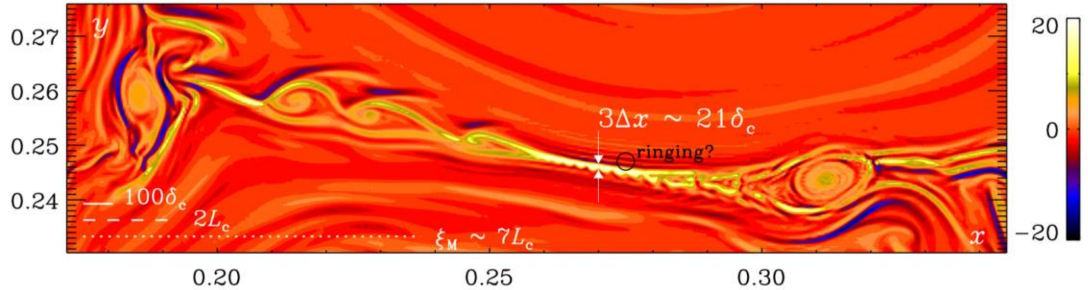
Determine C_{M} in relation:

$$\tau = C_{\mathrm{M}}\xi_{\mathrm{M}}/v_{\mathrm{A}}$$





3-D

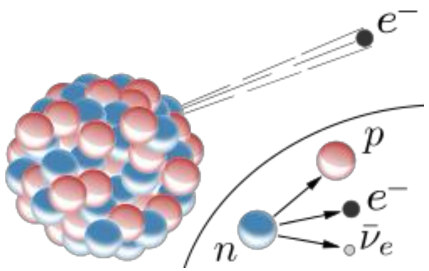


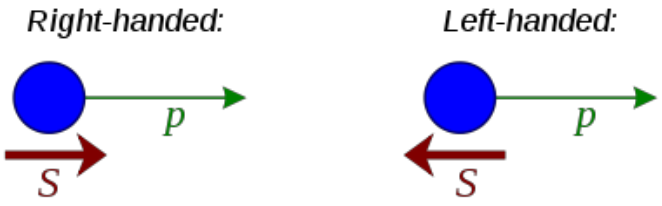
2-D

Chiral magnetic effect: introduces pseudoscalar

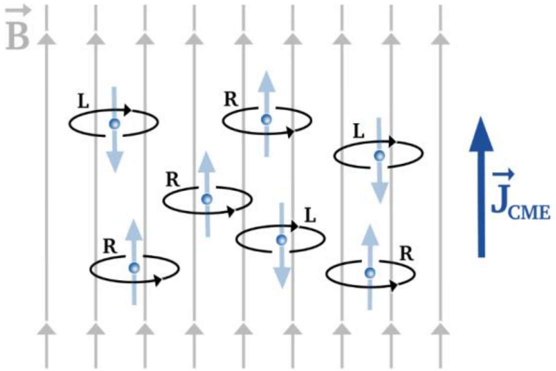
- Mathematically identical to α effect in mean-field dynamos
- Comes from chiral chemical potential μ (or μ_5)
- Number differences of left- & right-handed fermions

$$\mu_5 = 24\,\alpha_{\rm em}\,(n_{\rm L} - n_{\rm R})\,(\hbar c/k_{\rm B}T)^2,$$





- In the presence of a magnetic field, particles of opposite charge have momenta
- → electric current
- Self-excited dynamo
- But depletes μ



$$\frac{\partial \boldsymbol{A}}{\partial t} = \eta(\mu \boldsymbol{B} - \boldsymbol{\nabla} \times \boldsymbol{B}) + \boldsymbol{U} \times \boldsymbol{B}$$

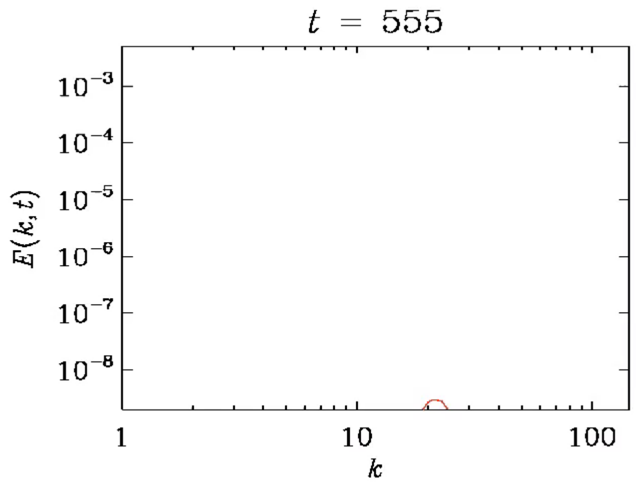
$$\sigma = |\mu k| - \eta k^2$$

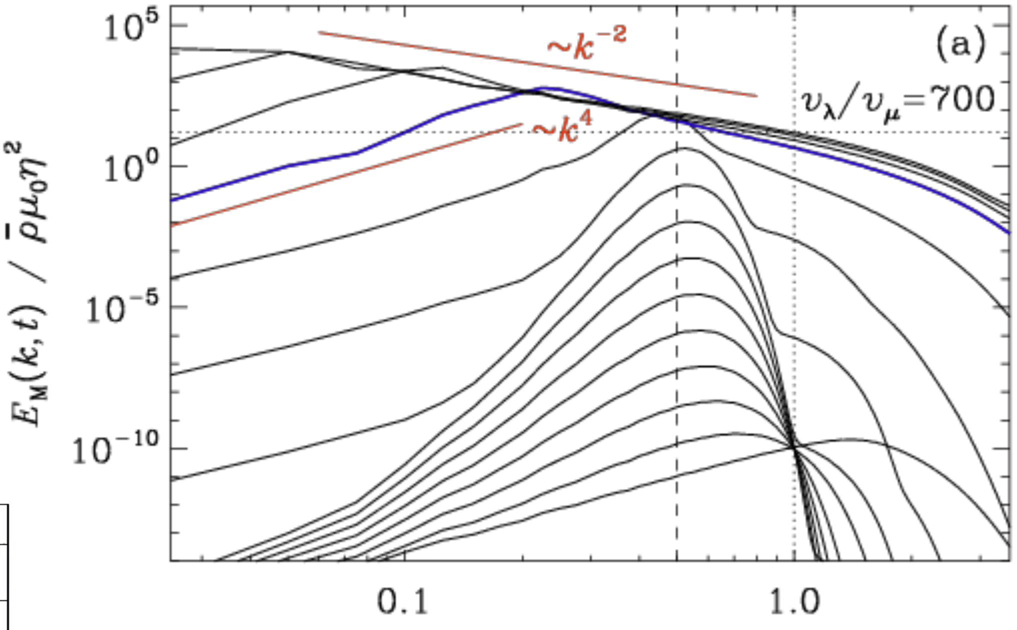
$\boldsymbol{B}=\mathrm{curl}\boldsymbol{A}$

Discovered originally by Vilenkin (1980); application to magnetogenesis in early Universe by Joyce & Shaposhnikov (1997)

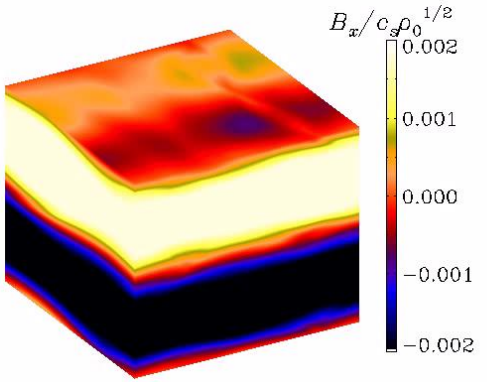
Time dependence from chiral magnetic effect (CME)

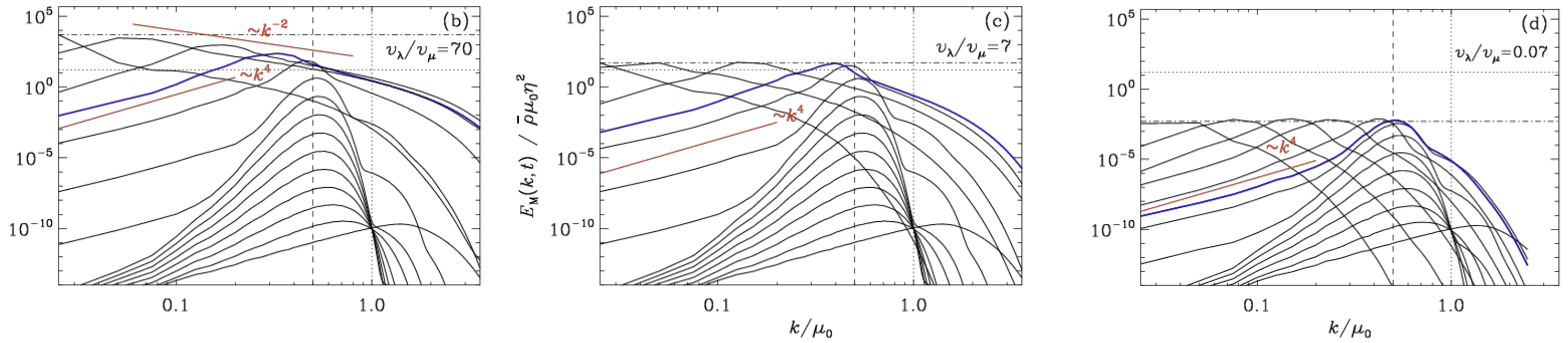
- Exponential growth at one k
- Subsequent inverse cascade
- Always fully helical





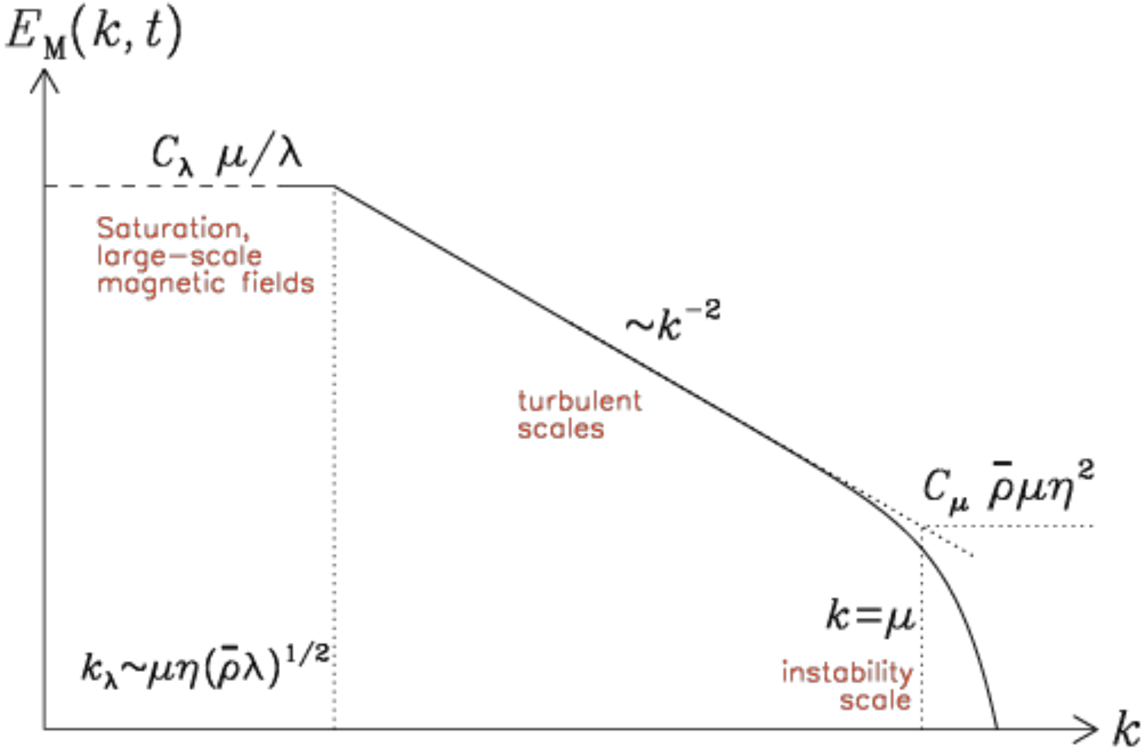
Growth at one wavenumber
Then: saturation caused by initial chemical potential





Brandenburg et al. (2017, ApJL 845, L21)

Many details are known by now



- Regime I is when turbulent subrange is long
- In regime II, just inverse cascading

$$\frac{\partial \boldsymbol{B}}{\partial t} = \boldsymbol{\nabla} \times [\boldsymbol{u} \times \boldsymbol{B} + \eta(\mu_5 \boldsymbol{B} - \boldsymbol{J})], \quad \boldsymbol{J} = \boldsymbol{\nabla} \times \boldsymbol{B},$$

$$\frac{\mathrm{D}\mu_5}{\mathrm{D}t} = -\lambda\, \eta\, (\mu_5 \boldsymbol{B} - \boldsymbol{J}) \cdot \boldsymbol{B} + D_5 \nabla^2 \mu_5 - \Gamma_{\mathrm{f}} \mu_5,$$

$$v_\lambda = \mu_{50}/\lambda^{1/2}, \qquad v_\mu = \mu_{50}\eta. \tag{6}$$

We recall that we have used here dimensionless quantities. We can identify two regimes of interest:

$$\eta k_1 < v_\mu < v_\lambda \quad \text{(regime I)}, \tag{7}$$

$$\eta k_1 < v_\lambda < v_\mu \quad \text{(regime II)}, \tag{8}$$

- Instability just η dependant
- Saturation governed by λ

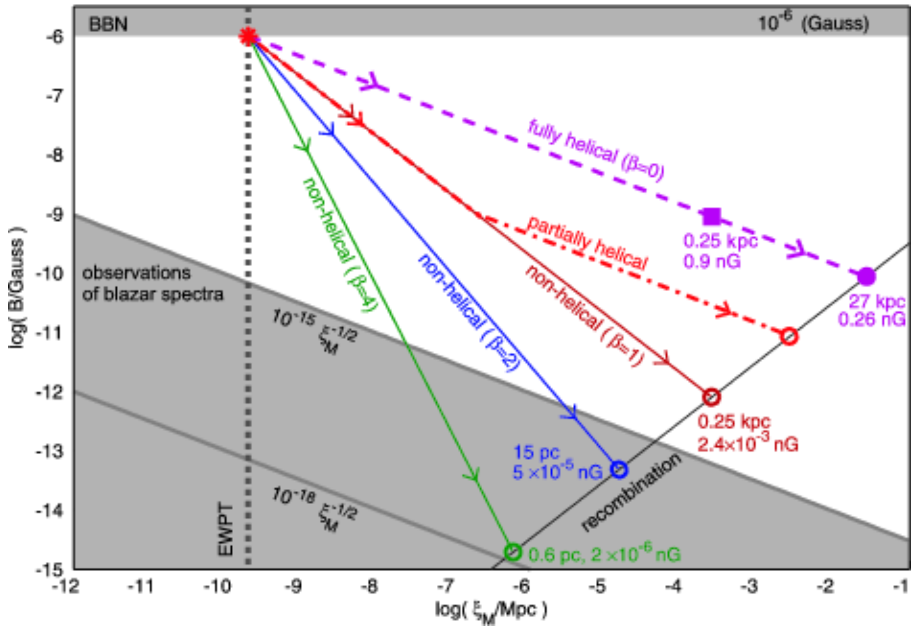
Strength of chiral magnetic effect

- Inverse turbulent cascade
  - $\langle B^2 \rangle \sim t^{-2/3}$ length scale: $\xi_M \sim t^{+2/3}$
- Dimensional arguments give

$$\langle \boldsymbol{B}^2 \rangle \, \xi_{\mathrm{M}} = \epsilon \, (k_{\mathrm{B}} T_0)^3 (\hbar c)^{-2},$$

- Inserting T=3K gives 10^{-18} G on 1 Mpc
- Consequence of conservation law

$$(n_{\mathrm{L}} - n_{\mathrm{R}}) + \frac{4\alpha_{\mathrm{em}}}{\hbar c} \langle \boldsymbol{A} \cdot \boldsymbol{B} \rangle = \mathrm{const.}$$



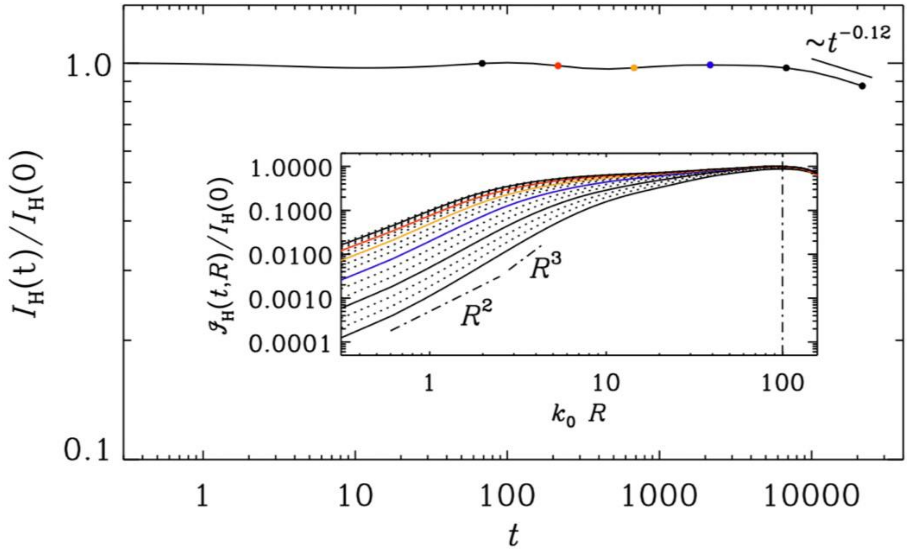
- But starting length scale very small → 12 cm
- Compared with horizon scale at that time (electroweak) of ~1 AU
- Other dimensional argument:

$$\langle \boldsymbol{B}^2 \rangle \xi_{\mathrm{M}} \lesssim \epsilon_3 (a_\star / a_0)^3 G^{-3/2} \hbar^{-1/2} c^{11/2},$$

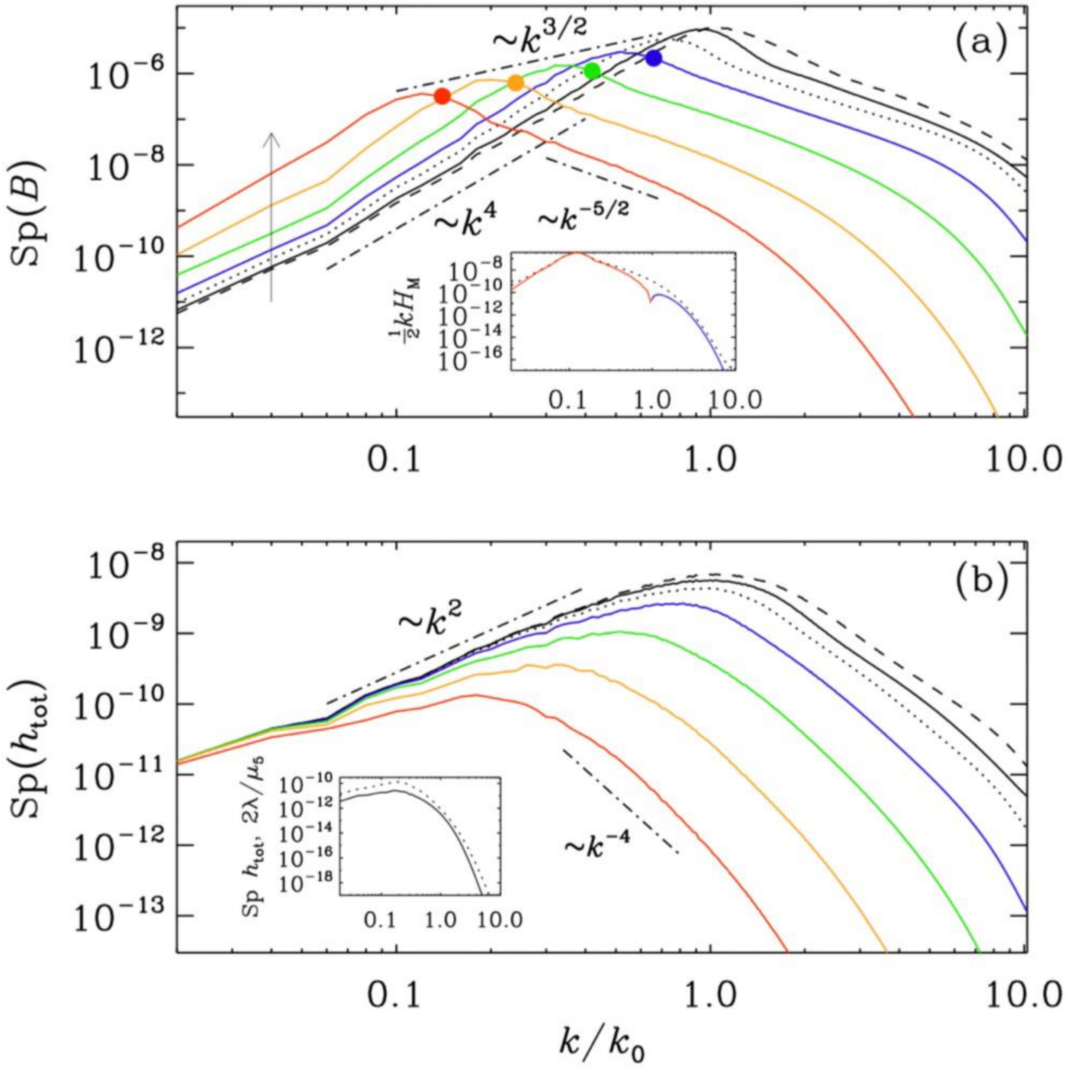
Decay law of magnetic turbulence with helicity balanced by chiral fermions

Axel Brandenburg,[1,2,3,4] Kohei Kamada,[5] and Jennifer Schober[6]

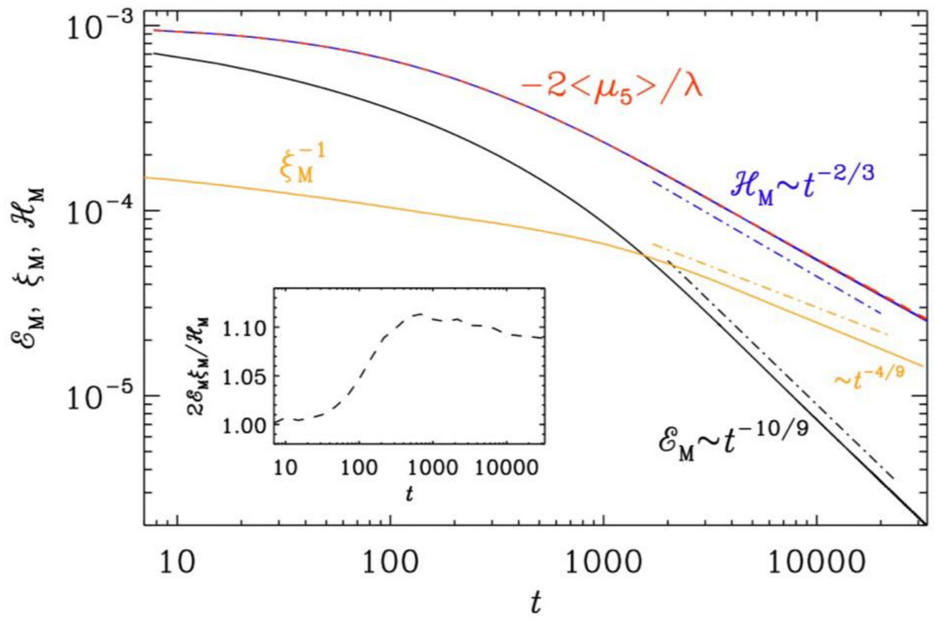
$$h_{\rm tot} \equiv \boldsymbol{A}\cdot\boldsymbol{B} + 2\mu_5/\lambda$$

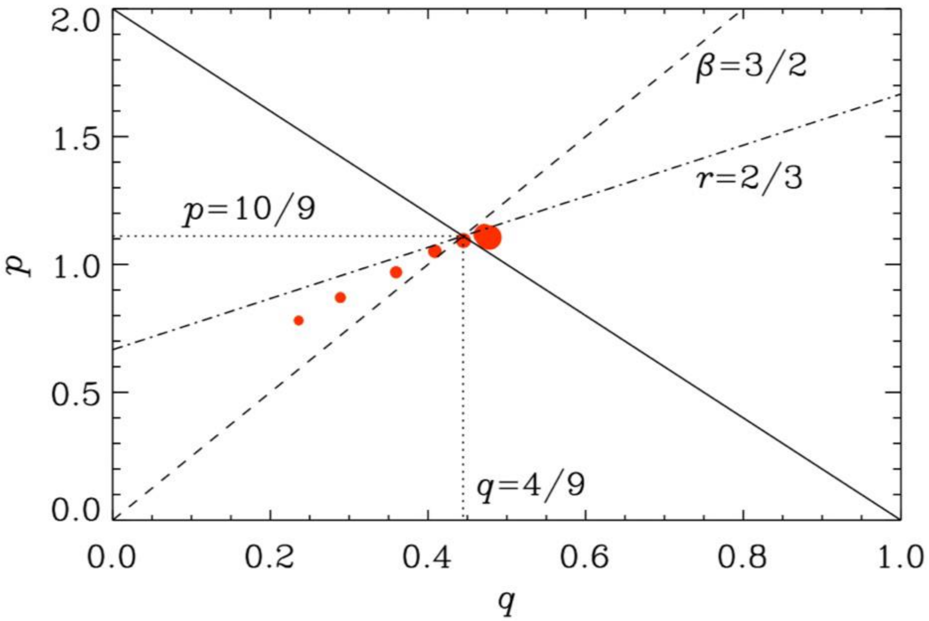


- Fully helical, but balanced chirality
- Hosking scaling applies
- Same scaling as for nonhelical turbulence
- But magnetic helicity not conserved: power law!



Algebraic decay of helicity





$$2\mathcal{E}_M\xi_M/\mathcal{H}_M \approx 1$$

$$|\mathcal{H}_M| \propto |\langle\mu_5\rangle| \propto t^{-r}$$

$$r = p - q = 2/3$$

- Helicity decays not exponentially,
- But algebraically: 10/9 – 4/9 = 6/9
- Important for baryogenesis